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Applications of Some Recent Techniques for the Exact Solutions of the Small Disturbance Potential Flow Equation of Nonequilibrium Transonic Gas Dynamics

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Abstract—Using the isovector approach as developed in [1] and the direct method of Clarkson and Kruskal [2], we have obtained similarity and other reductions to ordinary differential equations and exact solutions of the small disturbance potential flow equation in two dimension nonequilibrium transonic gas dynamics; namely,

$$K\left(C\phi_x - A\right)\phi_{xxx} + KC\phi_{xx}^2 + D\phi_x\phi_{xx} - K\phi_{xyy} - B\phi_{xx} - \phi_{yy} = 0.$$

Further, the Lie algebra of the transformation groups yielded by the isovector approach has enabled us to remark on its solvability and nilpotency.

Keywords—Isovector, Direct method, Similarity reductions, Lie algebra.

1. INTRODUCTION

Herein, we examine a class of flows governed by the small disturbance potential flow equation in two dimensions, from the realm of nonequilibrium transonic gas dynamics.

While studying the steady, inviscid, nonequilibrium flows, Vincenti [3], Moore and Gibson [4], and Clarke [5], using small perturbation theory, derived the following equation for the perturbation velocity potential ϕ, in the axisymmetric case:

$$K\frac{\partial}{\partial x}\left(A\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r}\right) + B\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} = 0. \tag{1.1}$$

In equation (1.1), K is a measure of the relaxation length,

$$A = 1 - M_{f_\infty}^2, \qquad B = 1 - M_{e_\infty}^2.$$

M_{f_∞} and M_{e_∞} are, respectively, the frozen and equilibrium Mach numbers based on the frozen and equilibrium speeds of sound.

Rhyming [6] presented a transonic correction to the above equation in which the zero term on the right-hand side is replaced by

$$KC\left(\phi_x\phi_{xx}\right)_x + D\phi_x\phi_{xx}, \tag{1.2}$$

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with

$$C = \frac{\left(1 - M_{f_\infty}^2\right) M_{f_\infty}^*}{\left(1 - M_{f_\infty}^*\right) U_\infty}, \qquad D = \frac{\left(1 - M_{e_\infty}^2\right) M_{e_\infty}^*}{\left(1 - M_{e_\infty}^*\right) U_\infty},$$

where $M_{f_\infty}^*$ and $M_{e_\infty}^*$ are the critical Mach numbers based on the frozen and equilibrium critical speeds of sound, respectively.

Sharma [7], using Bulman and Cole [8] procedure, considered a more general case

$$K\left(A\phi_{xxx} + \phi_{xzz}\right) + B\phi_{xx} + \phi_{yy} + \phi_{zz} - KC\left(\phi_x\phi_{xxx} + \phi_{xx}^2\right) - D\phi_x\phi_{xx} = 0. \tag{1.3}$$

However, on account of the heavy algebra involved in the calculations, no physical insight to the resulting solutions could be obtained. Thus, keeping in view the efficacy and limitations of the isovector [1,9,10] and 'direct method' [2,11] approaches *vis-a-vis* each other and other available techniques we have, herein, re-examined the said problem for the two-dimensional case via isovector and "direct method" techniques. The complete study is divided into two parts. Part 1 details the isovector approach, and the second part contains the details of "the direct method".

PART 1

2. ISOVECTOR APPROACH

For the case under consideration, equation (1.3) can be expressed as

$$K\left(C\phi_x - A\right)\phi_{xxx} + KC\phi_{xx}^2 + D\phi_x\phi_{xx} - K\phi_{xyy} - B\phi_{xx} - \phi_{yy} = 0. \tag{2.1}$$

For applying the isovector approach, we write equation (2.1) as a system of exterior one and two forms by introducing four new variables u, v, w, and s defined by

$$u = \phi_x, \qquad v = \phi_y, \tag{2.2}$$

$$w = \phi_{xx}, \qquad s = \phi_{xy} = u_y = v_x, \tag{2.3}$$

and

$$\alpha_1 = d\phi - u\,dx - v\,dy, \tag{2.4}$$

$$\alpha_2 = du - w\,dx - s\,dy, \tag{2.5}$$

$$\beta = (KCu - KA)\,dw \wedge dy + K\,ds \wedge dx + dv \wedge dx + \left(KC\,w^2 + Duw - Bw\right)\,dx \wedge dy. \tag{2.6}$$

In equations (2.4)–(2.6) α_1, α_2, and β are, respectively, 1, 1, and 2 forms, and the symbol $\wedge$ denotes the exterior product of differential forms.

Following the procedure outlined in [1], the exterior derivatives of α_1, α_2, and β can be expressed as

$$d\alpha_1 = -du \wedge dx - dv \wedge dy, \tag{2.7}$$

$$d\alpha_2 = -dw \wedge dx - ds \wedge dy, \tag{2.8}$$

and

$$d\beta = KC\,du \wedge dw \wedge dy + (2KCw + Du - B)\,dw \wedge dx \wedge dy + Dw\,du \wedge dx \wedge dy. \tag{2.9}$$

Closed Ideal[1]

Let $I = \{\alpha_1, \alpha_2, \beta, d\alpha_1, d\alpha_2\}$ be the fundamental ideal of the algebra of exterior forms $\wedge(E)$, where E is the manifold of dimension 7 in the space of variables x, y, ϕ, u, v, w, and s. The necessary and sufficient condition for I to be a closed ideal of the algebra of exterior differential forms is that $dI \subseteq I$. Alternatively, the exterior differential of a form in I is either contained in I or expressible as a linear combination of forms s in I. For the case under consideration, it is easily seen that the ideal $I\{\alpha_1, \alpha_2, \beta, d\alpha_1, d\alpha_2\}$ spanned by α_1, α_2, β, $d\alpha_1$, $d\alpha_2$ is a closed one.

Isovector Field

Define a vector field $\hat{V}$ over the space E_7 with components V^x, V^y, V^ϕ, V^u, V^v, V^w, V^s in the direction of x, y, ϕ, u, v, w, and s, respectively; that is,

$$V = V^x \frac{\partial}{\partial x} + V^y \frac{\partial}{\partial y} + V^\phi \frac{\partial}{\partial \phi} V^u \frac{\partial}{\partial u} + V^v \frac{\partial}{\partial v} + V^w \frac{\partial}{\partial w} + V^s \frac{\partial}{\partial s}, \tag{2.10}$$

where V^x, V^y, V^ϕ, V^u, V^v, V^w, V^s are functions of t, x, ϕ, u, v, w, and s.

A vectors field $\hat{V}$ is said to be an isovector field if

$$L_V I(\alpha_1, \alpha_2, \beta, d\alpha_1, d\alpha_2) \subseteq I(\alpha_1, \alpha_2, \beta, d\alpha_1, d\alpha_2), \tag{2.11}$$

where $L_V(\cdot)$ denotes the Lie derivative of $(\cdot)$ over the vector field V.

Transport Property of Forms α_1, α_2, and β

As mentioned in [1], for the calculation of isovectors we need to utilize the transport property. For the case under consideration, the transport property of exterior differential forms α, β can be expressed as

$$L_V(\alpha_1) = \lambda_1 \alpha_1, \tag{2.12}$$
$$L_V(\alpha_2) = \lambda_2 \alpha_2, \tag{2.13}$$
$$L_V(\beta) = \lambda_3 \beta + W_1 \wedge \alpha_1 + W_2 \wedge \alpha_2 - \lambda_4\, d\alpha_1 - \lambda_5\, d\alpha_2. \tag{2.14}$$

In equations (2.12)–(2.14), λ_1, λ_2, λ_3, λ_4, λ_5 are arbitrary functions of the variables x, y, ϕ, u, v, w, s, and W_1, W_2 are arbitrary 1-forms.

Further,

$$L_V(\alpha_i) = d(V \rfloor \alpha_i) + V \rfloor d\alpha_i \qquad (i = 1, 2), \tag{2.15a,b}$$
$$L_V(\beta) = d(V \rfloor \beta) + V \rfloor d\beta, \tag{2.16}$$

where d denotes the exterior differentiation and $\rfloor$ denotes inner multiplication of forms. Also, $d\beta$ occuring in (2.16) is a 3-form and is given in (2.9).

Calculation of Isovectors

Assume

$$V \rfloor \alpha_1 = \hat{G}(x, y, \phi, u, v, w, s), \tag{2.17}$$
$$V \rfloor \alpha_2 = \hat{H}(x, y, \phi, u, v, w, s), \tag{2.18}$$

[1]Importance of this property lies in the fact that it implies the integrability property of the forms in the solution manifold which is a submanifold of E defined under this subtitle.

and

$$W_1 = A_1\,dx + B_1\,dy + C_1\,d\phi + D_1\,du + E_1\,dv + F_1\,dw + G_1\,ds, \tag{2.19}$$
$$W_2 = A_2\,dx + B_2\,dy + C_2\,d\phi + D_2\,du + E_2\,dv + F_2\,dw + G_2\,ds, \tag{2.20}$$

where A_i, B_i, C_i, D_i, E_i, F_i, G_i $(i = 1, 2)$ are arbitrary functions of x, y, ϕ, u, v, w, s.

On combining equations (2.4), (2.6), (2.12), (2.17), (2.19), and (2.15a) and using equations (2.5), (2.6), (2.13), (2.18) in (2.15b) and performing expansion under exterior differentiation and inner multiplication operator and collecting the coefficients of similar 1-forms and equating them to zero and eliminating λ_1, λ_2 from the resulting equations, we arrive at the following system of partial differential equations involving V^x, V^y, V^ϕ, V^u, V^v, V^w, and V^s:

$$V^x = -\hat{H}_w, \tag{2.21a}$$
$$V^y = -\hat{H}_s = -\hat{G}_v, \tag{2.21b}$$
$$V^\phi = \hat{G} + uV^x + vV^y, \tag{2.21c}$$
$$V^u = \hat{G}_x + u\hat{G}_\phi + w\left(\hat{G}_u + V^x\right) = \hat{H} + wV^x + sV^y, \tag{2.21d}$$
$$V^v = \hat{G}_y + v\hat{G}_\phi + s\left(\hat{G}_u + V^x\right), \tag{2.21e}$$
$$V^w = \hat{H}_x + u\hat{H}_\phi + w\hat{H}_u, \tag{2.21f}$$
$$V^s = \hat{H}_y + v\hat{H}_\phi + s\hat{H}_u, \tag{2.21g}$$

and

$$\hat{G}_w = 0 = \hat{G}_s = \hat{H}_v. \tag{2.22}$$

On combining equations (2.21b), (2.21d), and (2.22), we get

$$\hat{G} = vY(y) + G(x, y, \phi, u), \tag{2.23}$$
$$\hat{H} = sY(y) + H(x, y, \phi, u, w), \tag{2.24}$$

where $Y(y)$ is an arbitrary function of y.

Using equations (2.23),(2.24) in (2.21a)–(2.21g), we get

$$\begin{aligned}
V^x &= -G_u,\\
V^y &= -Y(y),\\
V^\phi &= G - uG_u,\\
V^u &= G_x + uG_\phi,\\
V_v &= v\left(G_\phi + Y'\right) + G_y,\\
V^w &= G_{xx} + uG_{x\phi} + wG_{ux} + u\left(G_{x\phi} + G_{\phi\phi} + wG_{u\phi}\right) + w\left(G_{xu} + uG_{\phi u} + wG_{uu} + G_\phi\right),\\
V^s &= G_{xy} + sY' + uG_{y\phi} + wG_{uy} + v\left(G_{x\phi} + uG_{\phi\phi} + wG_{u\phi}\right) + s\left(G_{xu} + uG_{\phi u} + wG_{uu} + G_\phi\right).
\end{aligned} \tag{2.25}$$

From the transport property of $L_v(\beta)$ as represented through equation (2.16) and following the same procedure as outlined for $L_v(\alpha_i)$ and after detailed calculations, we arrive at the following system of equations:

$$V_y^x + vV_\phi^x + sV_u^x - V_w^s = 0, \tag{2.26}$$
$$V_s^s - V_v^v - KV_v^s = 0, \tag{2.27}$$
$$CV^u - (Cu - A)\left[V_s^s + V_x^x + uV_\phi^x + wV_u^x - V_w^w - V_y^y\right] = 0, \tag{2.28}$$

$$\begin{aligned} &V_y^y + vV_\phi^v + sV_u^v - DwV^u - s\left(V_y^x - vV_\phi^x + sV_u^x\right) \\ &+ \left(KCw^2 + DuW - Bw\right)\left(KV_v^s + V_v^v - Y_y^y\right) + K\left(vV_\phi^s + sV_u^s + V_y^s\right) \\ &- K(Cu - A)\left(uV_\phi^w + wV_u^w + V_x^w\right) - (2KCw + Du - B)V^w = 0. \end{aligned} \tag{2.29}$$

Equations (2.26) and (2.27), when translated in terms of G, yield

$$G = uX(x) + \frac{1}{K}\phi X(x) + \frac{1}{K}\phi Q(y) + R(x,y). \tag{2.30}$$

Equation (2.30), when combined with equation (2.28), yields

$$X(x) = -\frac{a}{K}, \qquad Q(y) = 2Y'(y) + \frac{a}{K}, \tag{2.31}$$

$$R(x,y) = -\frac{2A}{C}xY'(y) + f(y). \tag{2.32}$$

In equation (2.31), a is any arbitrary constant and $f(y)$ is an arbitrary function.

On combining equations (2.29), (2.31), and (2.32) we arrive at, for $b \neq 0$, $AD = BC$, and $b = 0$, the following two sets of values of the vectors components:

$$\begin{array}{ll} \text{for } b = 0, & \text{for } b \neq 0, \quad AD = BC, \\ V^x = a, & V^x = a, \\ V^y = -d_0, & V^y = -(by + d_0), \\ V^\phi = a_1 y + b_1, & V^\phi = 2b\left(\phi - \dfrac{A}{C}x\right) + a_1 y + b_1, \\ V^u = 0, & V^u = 2b\left(u - \dfrac{A}{C}\right), \\ V^v = a_1, & V^v = 3bv + a_1, \\ V^w = 0, & V^w = 2bw, \\ V^s = 0, & V^s = 3bs. \end{array} \tag{2.33a,b}$$

In equations (2.33a,b), a, b, d_0, a_1, and b_1 are arbitrary constants.

As it is known [10,11] that solvable Lie algebras obtained from infinitesimals of group transformations can be utilized to classify the similarity solutions of nonlinear partial differential equations, we, therefore, seek such information through the commutator table from which the structure constants can be easily seen. The basis vectors for the Lie algebra are

$$\begin{aligned} V^1 &= \partial_x, \\ V^2 &= -\partial_y, \\ V^3 &= \partial_\phi, \\ V^4 &= y\partial_\phi + \partial_v, \\ V^5 &= -y\partial_y + 2\left(\phi - \frac{A}{C}x\right)\partial_\phi + 2\left(u - \frac{A}{C}\right)\partial_u + 3v\partial_v + 2w\partial_w + 3s\partial_s. \end{aligned}$$

Vector V^5 ceases to exist for the case $b = 0$, as shown in Table 1.

Table 1. Commutator table for vectors (2.12).

	V^1	V^2	V^3	V^4	V^5
V^1	0	0	0	0	$-2\frac{A}{C}V^3$
V^2		0	0	$-V^3$	$-V^2$
V^3			0	0	$2V^3$
V^4				0	$3V^4$
V^5					0

Table 1 shows that this is a solvable Lie algebra. For $b = 0$, the algebra is nil-potent also, but it is not so for the case $b \neq 0$. Nevertheless, the optimal systems can still be calculated from the commutator table.

3. SIMILARITY TRANSFORMATIONS, SIMILARITY REDUCTIONS AND SIMILARITY SOLUTIONS

Herein, we utilize the isovector tabulated above for obtaining the orbital equations. Solutions of these said equations lead to the construction of similarity transformations, which when utilized in equation (2.1) reduces it to an ordinary differential equation of third order. The resulting nonlinear ordinary differential equation is either solved exactly or reduced to first order.

Moving on to similarity reductions, we notice that the condition $AD = BC$ is not physically realizable, and hence we confine our attention to the case $b = 0$. The vector field is now left with only four arbitrary constants and accordingly, we consider the following two cases.

CASE 1. $a_1 \neq 0$, $b = 0$.

In order to calculate the symmetries or similarity transformations, we need to solve the orbital equations. If the vector component corresponding to the variable x_i is V^{x_i}, then the orbital equation corresponding to it is

$$\frac{dx_i}{d\tilde{s}} = V^{x_i}. \tag{3.1}$$

The initial condition imposed on x_i is

$$x_i(0) = x^*. \tag{3.2}$$

Using the isovector listed in equation (2.33a), we can write

$$\frac{dx}{d\tilde{s}} = a, \qquad x(0) = x^*, \tag{3.3}$$

$$\frac{dy}{d\tilde{s}} = -d_0, \qquad y(0) = y^*, \tag{3.4}$$

$$\frac{d\phi}{d\underline{s}} = ay + b_1, \qquad \phi(0) = \phi^*. \tag{3.5}$$

The solutions of orbital equations (3.3)–(3.5) can be expressed as

$$x^* - x = a\tilde{s}, \tag{3.6}$$

$$y^* - y = -d_0\tilde{s}, \tag{3.7}$$

$$\phi^* - \phi = -\frac{a_1}{2d_0}\,(y^* - y)\,(y^* + y) - \frac{b_1}{d_0}\,(y^* - y)\,. \tag{3.8}$$

Thus, the similarity variables ξ and the new dependent variable U are given by

$$\xi = d_0 x + ay, \tag{3.9}$$

$$U(\xi) = \phi + \frac{a_1}{2d_0}y^2 + \frac{b_1}{d_0}y. \tag{3.10}$$

On using equations (3.9),(3.10) in (2.1), we get

$$KCd_0^4\left(U'U''' + U''^2\right) - Kd_0\left(Ad_0^2 + a^2\right)U'''' + Dd_0^3U'U''' - \left(Bd_0^2 + a^2\right)U'' + \frac{a_1}{d_0} = 0. \tag{3.11}$$

Integrating equation (3.11) with respect to ξ, we get

$$KCd_0^4U'U'' - Kd_0\left(Ad_0^2 + a^2\right)U'' + \frac{Dd_0^3}{2}U'^2 - \left(Bd_0^2 + a^2\right)U' + \frac{a_1}{d_0}\xi + k_0 = 0, \tag{3.12}$$

where k_0 is a constant of integration.

On assuming a solution of the form

$$U(\xi) = H_0 + G_0\xi + F_0\xi^n, \tag{3.13}$$

for equation (3.12), where H_0, G_0, F_0, and n are the constants to be determined, we find that

$$\begin{gathered} F_0 = \frac{2}{3d_0^2}\left(\frac{-2a_1}{D}\right)^{1/2}, \qquad G_0 = \frac{1}{d_0}\left(\frac{A-B}{C-D}\right), \quad H_0 \text{ arbitrary}, \\ \left(\frac{a^2}{d_0^2}\right) = \left(\frac{AD-BC}{C-D}\right), \qquad n = \frac{3}{2}, \qquad k_0 = \left[\frac{KCa_1}{D} + \frac{Dd_0^3}{2}\left(\frac{A-B}{C-D}\right)^2\right]. \end{gathered} \tag{3.14}$$

Thus, the exact solution to equation (2.1) can be expressed as

$$\phi(x,y) = \frac{2}{3d_0^2}\left(\frac{-2a_1}{D}\right)^{1/2}(d_0x+ay)^{3/2} + \left(\frac{A-B}{C-D}\right)(d_0x+ay) - \frac{a_1}{2d_0}y^2 - \frac{b_1}{d_0}y + H_0. \tag{3.15}$$

For further investigation of equation (3.12) we have, via the substitution $V = U'$, reduced it to the first-order equation in the standard Darboux form [14]

$$KCd_0^4VV' - Kd_0\left(Ad_0^2+a^2\right)V' + \frac{Dd_0^3}{2}V^2 - \left(Bd_0^2+a^2\right)V + \frac{a_1}{d_0}\xi + k_0 = 0. \tag{3.16}$$

CASE 2. $a_1 = 0 = b$.

For the case under consideration, the similarity variable ξ, the new dependent variable $U(\xi)$, and the reduced ordinary differential equation to equation (2.1) are given as under

$$\xi = d_0x + ay, \tag{3.17}$$

$$U(\xi) = \phi - \frac{b_1}{a}x, \tag{3.18}$$

$$\begin{aligned} KCd_0^4\left(U'U''' + U''^2\right) + K\left[\left(C\frac{b_1}{a} - A\right)d_0^3 - d_0\,a^2\right]U''' + Dd_0^3U'U'' \\ + \left[\left(\frac{Db_1}{a} - B\right)d_0^2 - a^2\right]U'' = 0. \end{aligned} \tag{3.19}$$

Integrating equation (3.19) with respect to ξ, we get

$$\begin{aligned} KCd_0^4U'U'' + K\left[\left(C\frac{b_1}{a} - A\right)d_0^3 - d_0\,a^2\right]U'' + D\frac{d_0^3}{2}U'^2 \\ + \left[\left(\frac{Db_1}{a} - B\right)d_0^2 - a^2\right]U' = C_0, \end{aligned} \tag{3.20}$$

where C_0 is a constant of integration.

Assuming $C_0 = 0$ and using the substitution

$$U' = p(U), \qquad UU'' = pp',$$

equation (3.20) can be reduced to the following form:

$$2d_0\frac{KC}{D}\left(\frac{p+k_1}{p+k_2}\right)dp = -dU, \tag{3.21}$$

where

$$
\begin{aligned}
k_1 &= \frac{1}{Cd_0^3}\left[\left(C\frac{b_1}{a} - A\right) d_0^2 - a^2\right], \\
k_2 &= \frac{2}{Dd_0^3}\left[\left(D\frac{b_1}{a} - A\right) d_0^2 - a^2\right].
\end{aligned} \tag{3.22}
$$

Equation (3.21), when integrated, yields

$$
\frac{2d_0KC}{D}\left[U' + (k_1 - k_2)\log(U' + k_2)\right] = -U + m, \tag{3.23}
$$

where m_0 is a constant of integration.

To the best of our knowledge, no closed form solutions of equation (3.23) are available. Nevertheless, equation (3.23) is a group theoretic reduction to the first-order ordinary differential equation.

Another reduction of equation (3.20) with $U' = V$ is

$$
K\left[Cd_0^4V + \left(C\frac{b_1}{a} - A\right) d_0^3 - d_0\, a^2\right] V' + \frac{Dd_0^3}{2} V^2 + \left[\left(\frac{Db_1}{a} - B\right) d_0^2 - a^2\right] V = C_0. \tag{3.24}
$$

Equation (3.24) is in the standard Darboux form. Rewriting equation (3.24), with k_1 and k_2 as defined in equation (3.22), we get

$$
\frac{d_0KC}{D}\left(\frac{2V + k_2 + (2k_1 - k_2)}{V^2 + k_2V - 2C_0/d_0^3D}\right) dV = -d\xi. \tag{3.25}
$$

Depending upon the sign of the expression $\lambda^2 = (2C_0/d_0^2D + b_0^2/4) \gtrless 0$, the integration of (3.25) yields

$$
\lambda^2 > 0, \qquad \frac{d_0kC}{D}\left[\log\left\{\left(U' + \frac{k_2}{2}\right)^2 - \lambda^2\right\} + \frac{(2k_1 - k_2)}{2\lambda}\log\frac{U' + k_2/2 - \lambda}{U' + k_2/2 + \lambda}\right] + \xi = C_3, \tag{3.26}
$$

$$
\lambda^2 < 0, \quad \frac{d_0kC}{D}\left[\log\left\{\left(U' + \frac{k_2}{2}\right)^2 - \lambda^2\right\} + \frac{(2k_1 - k_2)}{\lambda}\tan^{-1}\left(\frac{U' + k_2/2}{\lambda}\right)\right] + \xi = C_4. \tag{3.27}
$$

Equations (3.26),(3.27) can be handled for further reduction only for some special cases.

For the special case

$$
C\frac{b_1}{a} - A = \frac{a^2}{d_0^2}, \qquad C_0 = 0,
$$

equation (3.25) can be integrated twice to yield

$$
U = \frac{2}{d_0^3D}\left[\left(B - \frac{Db_1}{a}\right) d_0^2 + a^2\right]\xi + \frac{-2d_0KC}{D}k_5\, e^{-(D/2KCd_0)\xi} + k_6. \tag{3.28}
$$

In equation (3.28), k_5 and k_6 are constants of integration. Thus, the solution to equation (2.1) can be expressed as

$$
\phi = -\frac{b_1}{d_0}y + \frac{2}{d_0^3D}\left[\left(B - \frac{Db_1}{a}\right) d_0^2 + a^2\right](d_0x + ay) - \frac{2d_0KC}{D}k_5\, e^{-D(d_0x+ay)/2KCd_0} + k_6. \tag{3.29}
$$

Equation (3.29) represents another exact solution of equation (2.1) under the conditions $k_1 = 0$, $C_0 = 0$.

In the next section, we apply the direct approach of Clarkson and Kruskal [2], which does not utilize the group theory and has been interpreted in terms of nonclassical approach of finding infinitesimals of transformations.

PART 2

4. 'DIRECT METHOD' APPLIED TO EQUATION (2.1)

Following Clarkson and Kruskal [2], it is sufficient to seek similarity solution of equation (2.1) in the following special form:

$$\phi(x,y) = \alpha(x,y) + \beta(x,y)W\left(z(x,y)\right), \tag{4.1}$$

where $\alpha(x,y)$, $\beta(x,y)$, and $z(x,y)$ are assumed to be sufficiently differentiable functions and $W(z)$ two times differentiable.

Using equation (4.1) in equation (2.1) and collecting the coefficients of like derivatives and powers of $W(z)$, we get

$$\begin{aligned}
&KC\left(\alpha_x\alpha_{xxx} + \alpha_{xx}^2\right) - KA\alpha_{xxx} + D\alpha_x\alpha_{xx} - K\alpha_{xyy} - B\alpha_{xx} - \alpha_{yy}\\
&+W\Big[-\beta_{yy} - B\beta_{xx} - K\beta_{xyy} + D\left(\alpha_x\beta_{xx} + \alpha_{xx}\beta_x\right) + 2KC\alpha_{xx}\beta_{xx}\\
&\qquad - KA\beta_{xxx} + KC\left(\alpha_x\beta_{xxx} + \beta_x\alpha_{xxx}\right)\Big]\\
&+W'\Big[-2\beta_y z_y - \beta z_{yy} - B\left(2\beta_x z_x + \beta z_{xx}\right) + D\big\{\left(2\beta_x z_x + \beta z_{xx}\right)\alpha_x\\
&\qquad + \beta z_x z_{xx}\big\} - K\left\{\beta_{yy}z_x + 2\beta_{xy}z_y + 2\beta_y z_{xy} + \beta_x z_{yy} + \beta z_{xyy}\right\}\\
&\qquad + 2KC\alpha_{xx}\left(2\beta_x z_x + \beta z_{xx}\right) - KA\left(3\beta_{xx}z_x + 3\beta_x z_{xx} + \beta z_{xxx}\right)\\
&\qquad + KC\left\{\alpha_x\left(3\beta_{xx}z_x + 3\beta_x z_{xx} + \beta z_{xxx}\right) + \beta z_x\alpha_{xxx}\right\}\Big]\\
&+W''\Big[-\beta z_y^2 - B\beta z_x^2 - K\left(2\beta_y z_x z_y + \beta z_x z_{yy} + \beta_x z_y^2 + 2\beta z_y z_{xy}\right)\\
&\qquad + D\alpha_x\beta z_x^2 + 2KC\beta\alpha_{xx}z_x^2 - KA\left(3\beta_x z_x^2 + 3\beta z_x z_{xx}\right)\\
&\qquad + KC\alpha_x\left(3\beta_x z_x^2 + 3\beta z_x z_{xx}\right)\Big]\\
&+W'''\left[-K\beta z_x z_y^2 - KA\beta z_x^3 + KC\beta\alpha_x z_x^3\right]\\
&+W^2\left[D\beta_x\beta_{xx} + KC\beta_{xx}^2 + KC\beta_x\beta_{xxx}\right]\\
&+WW'\Big[D\left\{\beta_x\left(2\beta_x z_x + \beta z_{xx}\right) + \beta z_x\beta_{xx}\right\} + 2KC\beta_{xx}\left(2\beta_x z_x + \beta z_{xx}\right)\\
&\qquad + KC\left\{\beta_x\left(3\beta_{xx}z_x + 3\beta_x z_{xx} + \beta z_{xxx}\right) + \beta z_x\beta_{xxx}\right\}\Big]\\
&+WW''\left[D\beta\beta_x z_x^2 + KC\left\{2\beta\beta_{xx}z_x^2 + 3\beta_x z_x\left(\beta_x z_x + \beta z_{xx}\right)\right\}\right]\\
&+W'W''\left[2KC\beta z_x^2\left(2\beta_x z_x + \beta z_{xx}\right) + 3KC\beta z_x\left(\beta_x z_x^2 + \beta z_x z_{xx}\right) + D\beta^2 z_x^3\right]\\
&+W'^2\Big[D\beta z_x\left(2\beta_x z_x + \beta z_{xx}\right) + KC\left(4\beta_x^2 z_x^2 + \beta^2 z_{xx}^2 + 4\beta\beta_x z_x z_{xx}\right)\\
&\qquad + KC\beta z_x\left(3\beta_{xx}z_x + 3\beta_x z_{xx} + \beta z_{xxx}\right)\Big]\\
&+KC\beta^2 z_x^4\left(W''^2 + W'W'''\right) + WW'''\left(KC\beta\beta_x z_x^3\right) = 0.
\end{aligned} \tag{4.2}$$

In order that equation (4.2) be transformed into an ordinary differential equation for $W(z)$, it is necessary that ratios for different derivatives and powers of $W(z)$ must be functions of z only.

Choosing the coefficient of $(W''^2 + W'W'')$ as the normalizing coefficient, we express the rest of the coefficients in terms of products of $(\beta^2 z_x^4)$ and some functions of z, which are to be determined. From the coefficient of WW''', we get

$$\beta\beta_x z_x^3 = \Gamma_1(z)\beta^2 z_x^4, \tag{4.3}$$

where $\Gamma_1(z)$ is a function to be determined, using the freedoms mentioned in Remark A.4.3b; see the Appendix.

From equation (4.3),

$$\frac{\beta_x}{\beta} = \Gamma_1(z)\, z_x.$$

On integrating w.r.t. x,

$$\beta = \Gamma_1(z) F(y),$$

where $F(y)$ is an arbitrary function of y.

By Remark A.4.3b,

$$\Gamma_1(z) = 1,$$

which implies that

$$\beta = \beta(y). \tag{4.4a,b}$$

Similarly, from the coefficient of W'^2, we have

$$D z_x z_{xx} + KC\left(z_{xx}^2 + z_x z_{xxx}\right) = \Gamma_2(z) z_x^4. \tag{4.5}$$

From the coefficient of $W'W''$, one can write

$$5KC\,\frac{z_{xx}}{z_x} + D = \Gamma_3(z) z_x. \tag{4.6}$$

Integrating equation (4.6) twice, w.r.t. x and using the freedom for the determination of z, we get

$$\Gamma_3(z) = \frac{-5KC}{D}\, \mathrm{e}^{-(D/5KC)x + \ln g^1(y)} + g^2(y).$$

Thus,

$$z = \frac{-5KC}{D}\, g^1(y)\, \mathrm{e}^{-(D/5KC)x} + g^2(y), \tag{4.7}$$

where $g^1(y)$, $g^2(y)$ are arbitrary functions of y.

Using equation (4.7) in equation (4.5), and using Remark A.4.3c and then comparing with (4.7), we have

$$g^2(y) = \text{constant} = c_1,$$

and

$$\Gamma_2(z) = \frac{1}{(z - c_1)^2}. \tag{4.8}$$

From the coefficient of W''' and Remark A.4.3a, we get

$$\alpha = \frac{1}{KC} g^3(y) + \frac{1}{C}\left[A + \left(\frac{25k^2C^2}{D^2}\right)\left(\frac{g^{1\prime}}{g^1}\right)^2\right] x. \tag{4.9}$$

Further, from the term independent of W, we can write

$$-\frac{1}{KC} g''^3(y) - \frac{25K^2C}{D^2}\left\{\left(\frac{g'^1}{g^1}\right)^2\right\}'' x - \frac{25K^3C}{D^2}\left\{\left(\frac{g'^1}{g^1}\right)^2\right\}'' = \Gamma_5(z)\beta^2 z_x^4. \tag{4.10}$$

Comparing the terms containing x, we conclude

$$\left[\left(\frac{g^{1\prime}}{g^1}\right)^2\right]'' = 0,$$

which has a solution

$$g^1 = c_4 e^{c_5 y}.$$

The terms depending upon y yield

$$g^{3''}(y) = 0, \qquad \Gamma_5(z) = 0,$$

which, in turn, imply

$$z = -\frac{5KC}{D} c_4 \, e^{(c_5 y - (D/5KC)x)} + c_1. \tag{4.11}$$

From the coefficient of W, we get

$$\frac{\beta''}{\beta^2} = \Gamma_6(z) \left\{ (c_1 - z) \frac{D}{5KC} \right\}^4.$$

Assuming $\beta''/\beta^2 = (3/2)c_7^2$, and integrating, we get

$$\beta^{2\prime} = c_7^2 \beta^3 + c_8^2. \tag{4.12}$$

We consider two cases here with the coefficient of W'.

CASE 1. $c_7 = 0$, i.e., $\beta = c_8 y + c_9$, which yields $c_8 = 0$, i.e., $\beta = \beta_0$, a constant.

CASE 2. $c_8 = 0$, $\beta = 4(c_7 y + c_9)^{-2}$, which implies $c_7 = 0$, and hence again $\beta = \beta_0$.

Accordingly, the coefficient of W' provides

$$\Gamma_7(z)\beta_0 \left(\frac{D}{5KC}\right)^4 (z - c_1)^3 = \left(\frac{D-C}{C}\right) c_5^2 + \left(\frac{D}{5KC}\right)^2 \left(\frac{AD - BC}{C}\right), \tag{4.13}$$

and the coefficient of W'' yields

$$\Gamma_8(z)\beta_0 \left(\frac{D}{5KC}\right)^4 (z - c_1)^2 = \left(\frac{D-C}{C}\right) c_5^2 + \left(\frac{D}{5KC}\right)^2 \left(\frac{AD - BC}{C}\right). \tag{4.14}$$

Thus, equation (4.2) gets reduced to

$$KC \left(W'W''' + W''^2\right) \frac{-3KC}{(z - c_1)^2} W'^2 + \frac{M}{(z - c_1)^3} W' + \frac{M}{(z - c_1)^2} W'' = 0, \tag{4.15}$$

where

$$\begin{aligned}
M &= \frac{1}{\beta_0} \left(\frac{5KC}{D}\right)^4 \left[\left(\frac{D-C}{C}\right) c_5^2 + \left(\frac{D}{5KC}\right)^2 \left(\frac{AD - BC}{C}\right)\right] \\
z &= -\frac{5KC}{D} c_4 \, e^{(c_5 y - (D/5KC)x)} + c_1, \\
\alpha &= \frac{1}{KC} (c_2 y + c_3) + \frac{1}{C} \left(A + \frac{25K^2C^2}{D^2} c_5^2\right) x, \\
\beta &= \beta_0.
\end{aligned} \tag{4.16}$$

Any solution of equation (4.15) gives a solution of equation (2.1) by virtue of the relation

$$\phi = \alpha + \beta W.$$

Now, we move on to seeking solutions of equation (4.15) under certain assumptions.

CASE 1. $M = 0$, i.e.,

$$c_5^2 = \frac{(AD - BC)D^2}{25K^2C^2(C - D)}.$$

The solution is

$$W = a\,(z - c_1)^{5/2},$$

which yields

$$\phi = \frac{c_3}{KC} + \left(\frac{A-B}{C-D}\right) x + \frac{c_2}{KC} y + c_9 \, e^{((5/2)c_5 y - (D/2KC)x)}. \tag{4.17}$$

CASE 2. Writing Z for $(z - c_1)$ and assuming $W'' = -W'/Z$, equation (4.15) transforms to

$$(W'W'')' = \frac{3}{Z^2} W'^2. \tag{4.18a}$$

Integrating equation (4.18a), the solution is

$$W = B_0 \ln Z + k_0, \tag{4.18b}$$

which yields a trivial solution.

CASE 3. $M = mKC$, m is an arbitrary constant.

For this case, equation (4.15) under the transformation $Z = e^Y$ assumes the following form:

$$\dot{W}\ddot{W} - \dot{W}\dddot{W} + \ddot{W}^2 + m\ddot{W} = 0. \tag{4.19}$$

A solution to equation (4.19) is

$$W = \frac{2}{5} m \ln Z + a_1 Z^{5/2} + a_0, \tag{4.20}$$

which is similar to (4.17).

5. CONCLUSIONS

Following the isovector approach and the 'direct method' of Clarkson and Kruskal, we have obtained for equation (2.1) the similarity reductions (see equations (3.11) and (3.19)) and other (equations (4.15) and (4.19)) to ordinary differential equations (ODEs) of third order. Further, through standard procedure the said ODEs have either been further reduced to first-order standard Darboux form (equations (3.16) and (3.24)) and other (see equation (3.23)), or solved completely. The notable exact solutions are presented through equations (3.15), (3.29), and (4.17).

APPENDIX

In order to put the direct approach into practice, we have to follow the remarks recorded below (for details see [2]).

REMARK A.4.1. We shall use the coefficient of highest derivatives of $w(z)$ as the normalising coefficient and require that other coefficients are of the form of the normalising coefficient multiplied by $\Gamma(z)$, where Γ is a function of z to be determined.

REMARK A.4.2. Whenever we use an upper-case Greek letter to denote a function (e.g., $\Gamma(z)$), then this is a function, to be determined, upon which we can perform any mathematical operation (e.g., differentiation, integration, taking logarithm, exponentiation, taking powers, rescaling, etc.) and then also call the resulting function $\Gamma(z)$, without loss of generality (e.g., the differential of $\Gamma(z)$ will be $\Gamma(z)$).

REMARK A.4.3. There are three freedoms in the determination of α, β, z which we can exploit, without loss of generality:

(a) if $\alpha(x,t)$ is of the form $\alpha(x,t) = \alpha_0(x,t) + \beta(x,t)\Gamma(z)$, where $\alpha_0(x,t)$ is specified and $\Gamma(z)$ is any function, then we can assume that $\Gamma \equiv 0$ (make the transformation $w(z) \to w(z) - \Gamma(z)$);

(b) if $\beta(x,t)$ is of the form $\beta = \beta_0(x,t)\Gamma(z)$, where β_0 is specified and $\Gamma(z)$ is any function, then we can assume that $\Gamma \equiv 1$ (make the transformation $w(z) \to w(z)/\Gamma(z)$);

(c) if $z(x.t)$ is defined by an equation of the form $\Gamma(z) = z_0(x.t)$, where z_0 is specified and $\Gamma(z)$ is any invertible function, then we can assume that $\Gamma z = z$ (make the transformation $z \to \Gamma^{-1}(z)$ where Γ^{-1} is the inverse of Γ).

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